SEM-DG Approximation for elasto-acoustics

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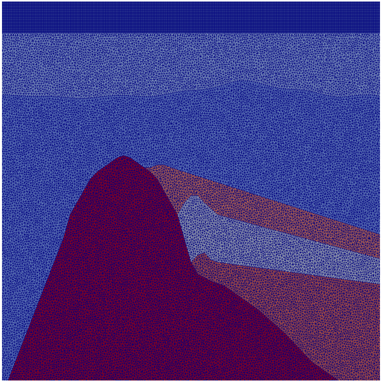
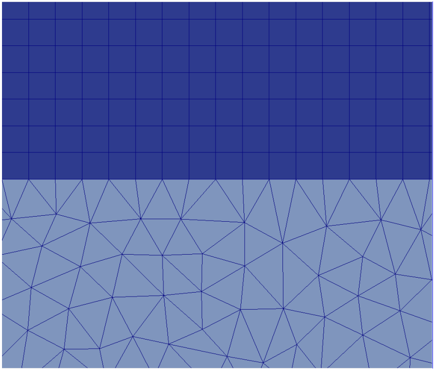
Why using hybrid meshes?





- Useful when the use of unstructured grid is non-sense (e.g. medium with a layer of water).
- Well suited for the coupling of numerical methods in order to reduce the computational cost and improve the accuracy.

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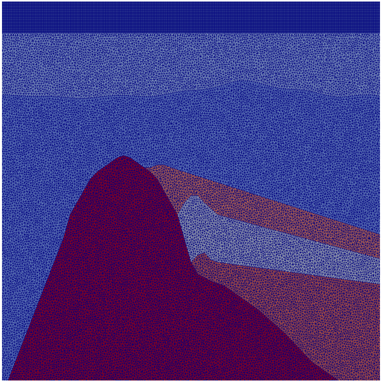

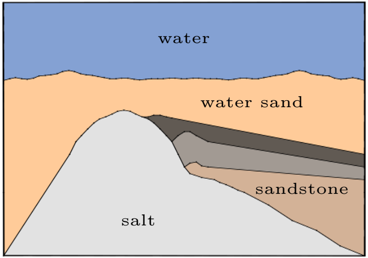

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Elastodynamic system

$x \in \Omega \subset \mathbb{R}^d$, $t \in [0, T]$, $T > 0$:

$$\begin{cases} \rho(x)\dfrac{\partial v}{\partial t}(x,t) & = \nabla \cdot \underline{\underline{\sigma}}(x,t), \\ \dfrac{\partial \underline{\underline{\sigma}}}{\partial t}(x,t) & = \underline{\underline{C}}(x)\underline{\underline{\epsilon}}(v(x,t)). \end{cases}$$

With:

- $\rho(x)$ the density,
- $\underline{\underline{C}}(x)$ the elasticity tensor,
- $\underline{\underline{\epsilon}}(x,t)$ the deformation tensor,
- $v(x,t)$, the wavespeed,
- $\underline{\underline{\sigma}}(x,t)$ the strain tensor.

Software written in **Fortran** for wave propagation simulation in the **time domain**

Features

Simulation:

- on various types of meshes (**unstructured triangles and tetrahedra**),
- on **heterogeneous** media (**acoustic, elastic and elasto-acoustic**).

- **Discontinuous Galerkin** (DG) based on **unstructured triangles and unstructured tetrahedra**,
- with various time-schemes : **Runge-Kutta (2 or 4), Leap-Frog**,
- with **multi-order** computation(**p-adaptivity**)...

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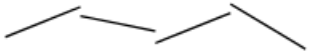
1. Numerical Methods
    - Discontinous Galerkin Method (DG)
    - Spectral Element Method (SEM)
    - Advantages of each method

Discontinuous Galerkin Method

Use discontinuous functions :

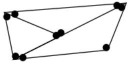
mesh

continuous



discontinuous

h adaptivity :



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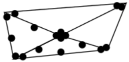
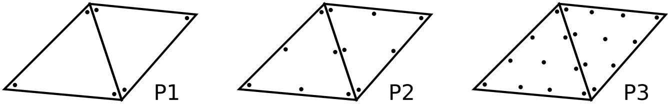
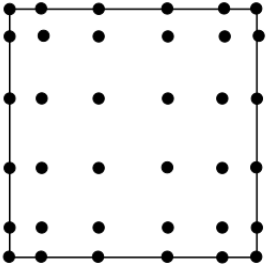


Degrees of freedom necessary on each cell :



General principle

- Finite Element Method (FEM) discretization + Gauss-Lobatto quadrature,
- Gauss-Lobatto points as degrees of freedom (gives us exponential convergence on L^2-norm).



- $\displaystyle\int f(x)dx \approx \sum_{j=1}^{N+1} \omega_j f(\xi_j)$,
- $\varphi_i(\xi_j) = \delta_{ij}$.

Advantages of each method

DG

- Element per element computation (*hp*-adaptivity).
- Time discretization quasi explicit (block diagonal mass matrix).
- Simple to parallelize.
- Robust to brutal changes of physics and geometry

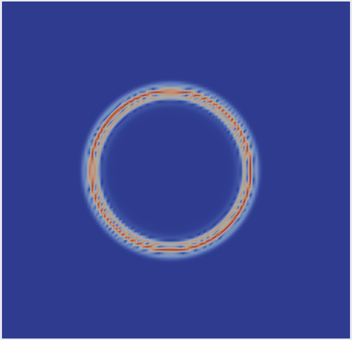
SEM

- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method.
- Simplifies the mass and stiffness matrices (mass matrix diagonal).

2. Comparison DG/SEM on structured quadrangle mesh
    - Description of the test cases
    - Comparative tables

Description of the test cases

Physical parameters



P wavespeed	$1000\ m.s^{-1}$
Density	$1\ kg.m^{-3}$

Second order **Ricker Source** in *Pwave* ($f_{peak} = 10Hz$)

General context

- **Acoustic homogeneous** medium.
- Four different meshes : **10000 cells, 22500 cells**, 90000 cells, 250000 cells.
- CFL computed using **power iteration** method.
- **Leap-Frog** time scheme.
- **Eight threads** parallel execution with **OpenMP**.

Comparative tables

- Error computed as the difference between an analytical and a numerical solution for each method.
- Three cases considered : DG without penalization terms, DG with penalization terms and SEM.

	CFL	L2-error	CPU-time	Nb of time steps
DG($\alpha = 0$)	3.18e-3	2e-1	5.13	629
SEM	4.9e-3	5e-2	0.80	409

Figure: DG not penalized and SEM comparison on the 10000 cells case

	CFL	L2-error	CPU-time(s)	Nb of time steps
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Figure: DG not penalized and SEM comparison on the 20000 cells case

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	CFL	L2-error	CPU-time(s)	Nb of time steps
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Figure: DG penalized and SEM comparison using the same CFL on a 10000 thousands cells mesh

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DG ($\alpha = 0.5$)	1.33e-3	2e-2	32.98	1502
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Advantages of each method

DG

- Element per element computation (*hp*-adaptivity).
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SEM

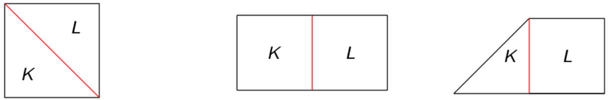
- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method.
- Simplifies the mass and stiffness matrices (mass matrix diagonal).
- **Reduces the computational costs on structured quadrangle cells in comparison with DG**

3. DG/SEM coupling
    - Hybrid meshes structures
    - Variational formulation

Hybrid meshes structures

- Aim at coupling P_k and Q_k structures.
- Need to extend or split some structures (e.g. neighbour indexes).
- Define new face matrices:

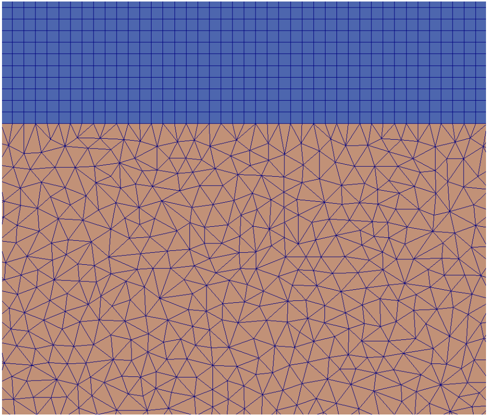
$$M_{ij}^{K,L} = \int_{K\cap L} \phi_i^K \phi_j^L, \quad M_{ij}^{K,L} = \int_{K\cap L} \psi_i^K \psi_j^L, \quad M_{ij}^{K,L} = \int_{K\cap L} \phi_i^K \psi_j^L.$$



Variational formulation

Global context

- Domain in two parts : $\Omega_{h,1}$ (**structured quadrangles + SEM**), $\Omega_{h,2}$ (**unstructured triangles + DG**).



Variational formulation

SEM variational formulation :

$$\begin{cases} \displaystyle\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \int_{\Gamma_{out,1}} (\sigma_1 \mathbf{n}_1) \cdot w_1, \\ \displaystyle\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = -\int_{\Omega_{h,1}} (\nabla(C\xi_1)) \cdot v_1 + \int_{\Gamma_{out,1}} (C\xi_1 \mathbf{n}_1) \cdot v_1. \end{cases}$$

DG variational formulation :

$$\begin{cases} \displaystyle\int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = -\int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{out,2}} (\sigma_2 \mathbf{n}_2) \cdot w_2 + \int_{\Gamma_{int}} \{\{\sigma_2\}\}[[w_2]] \cdot \mathbf{n}_2, \\ \displaystyle\int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = -\int_{\Omega_{h,2}} (\nabla(C\xi_2)) \cdot v_2 + \int_{\Gamma_{out,2}} (C\xi_2 \mathbf{n}_2) \cdot v_2 + \int_{\Gamma_{int}} \{\{v_2\}\}[[C\xi_2]] \cdot \mathbf{n}_2. \end{cases}$$

Variational formulation

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\begin{cases}
\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 + \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 \\
+ \int_{\Gamma_{out,1}} (\sigma_1 \mathbf{n}_1) \cdot w_1 + \int_{\Gamma_{out,2}} (\sigma_2 \mathbf{n}_2) \cdot w_2 + \int_{\Gamma_{int}} \{\{\sigma_2\}\}[[w_2]] \cdot \mathbf{n}_2 \\
+ \int_{\Gamma_{1/2}} [[\sigma w]] \cdot \mathbf{n}, \\
\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 + \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = -\int_{\Omega_{h,1}} (\nabla (C\xi_1)) \cdot v_1 - \int_{\Omega_{h,2}} (\nabla (C\xi_2)) \cdot v_2 \\
+ \int_{\Gamma_{out,1}} (C\xi_1 \mathbf{n}_1) \cdot v_1 + \int_{\Gamma_{out,2}} (C\xi_2 \mathbf{n}_2) \cdot v_2 + \int_{\Gamma_{int}} \{\{\sigma_2\}\}[[w_2]] \cdot \mathbf{n}_2 \\
+ \int_{\Gamma_{1/2}} [[(C\xi) v]] \cdot \mathbf{n}.
\end{cases}
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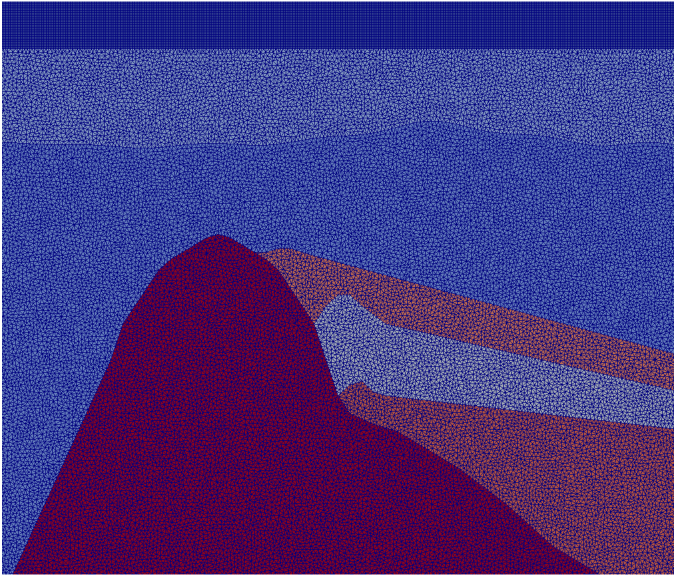

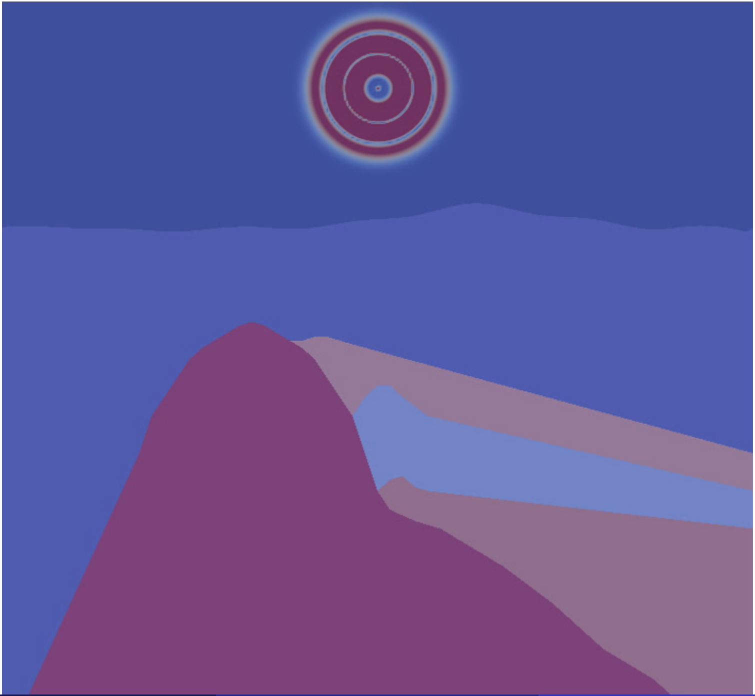

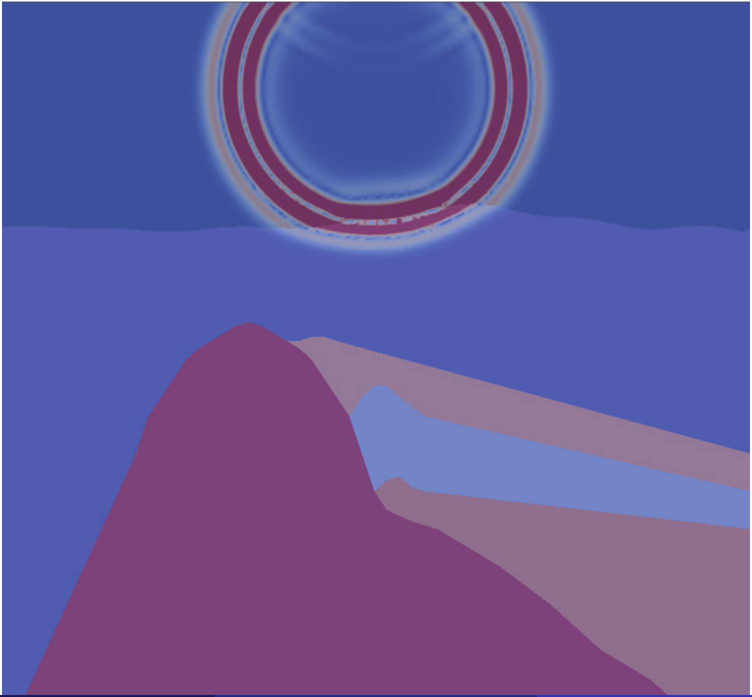

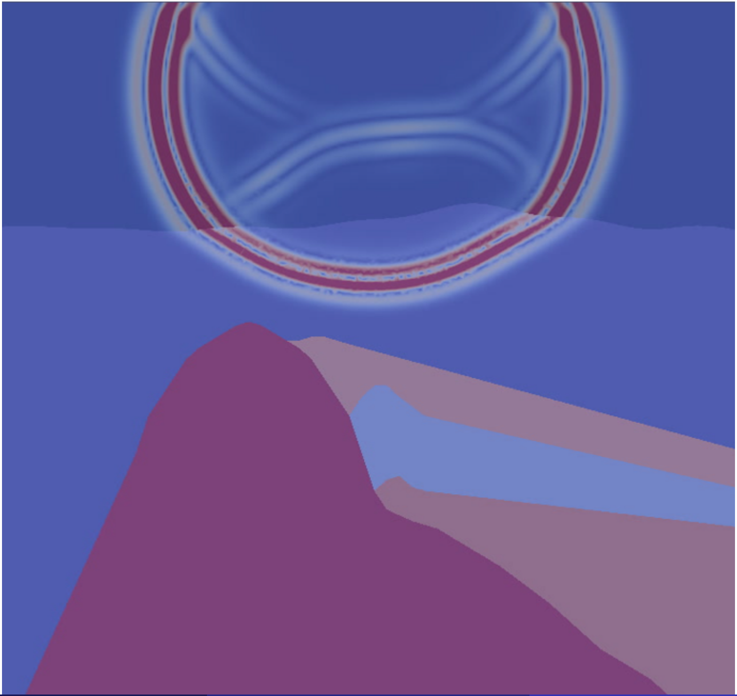

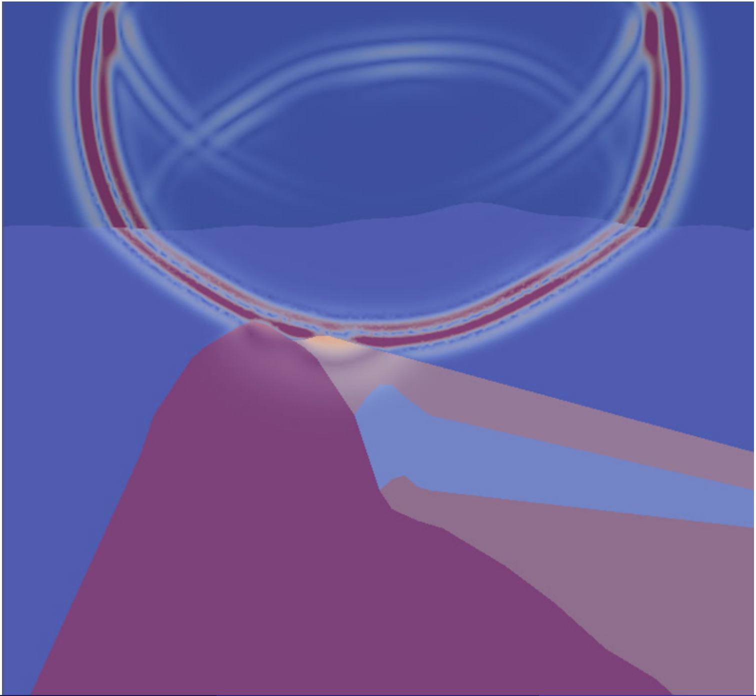

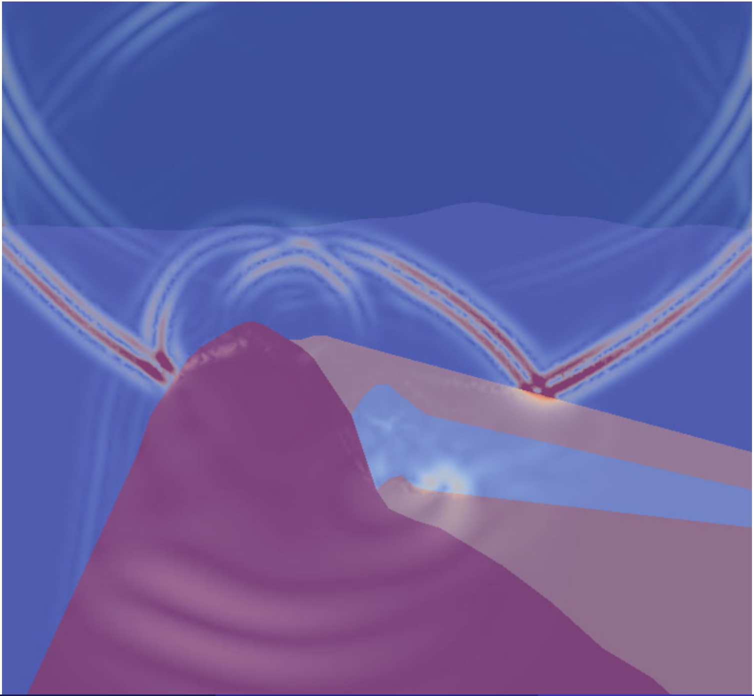

General settings

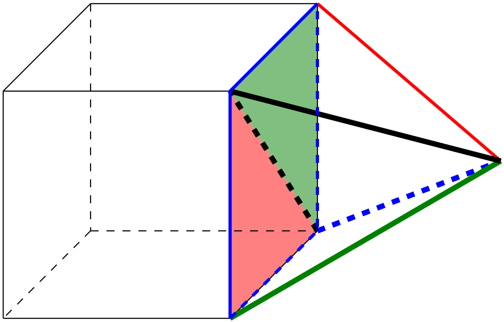

Figure: Hexa/Tet boundary configuration

- Only deal with a simple case of 3D hybrid meshes : one hexahedron has only two tetrahedra as neighbour.
- Extend SEM in 3D (basis functions...).
- Require introducing a new matrix which handles the rotation cases between two elements.

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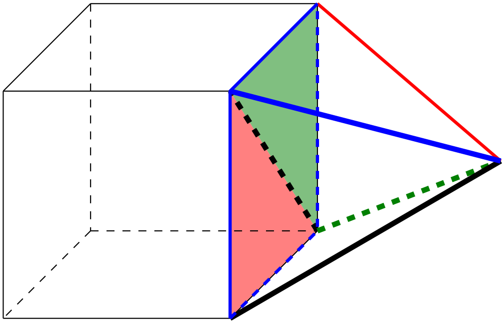

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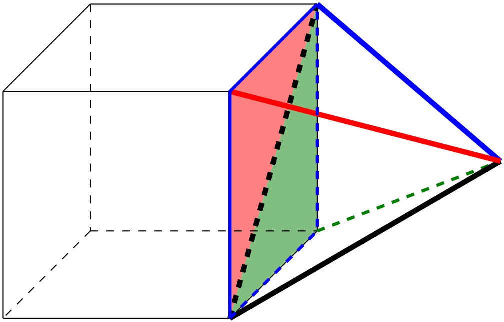

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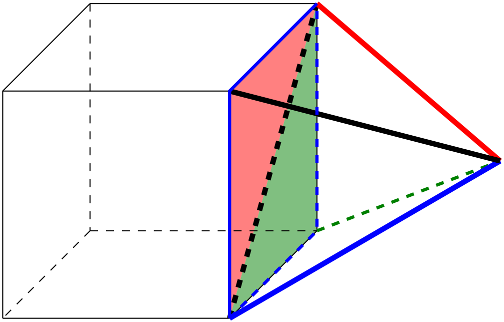

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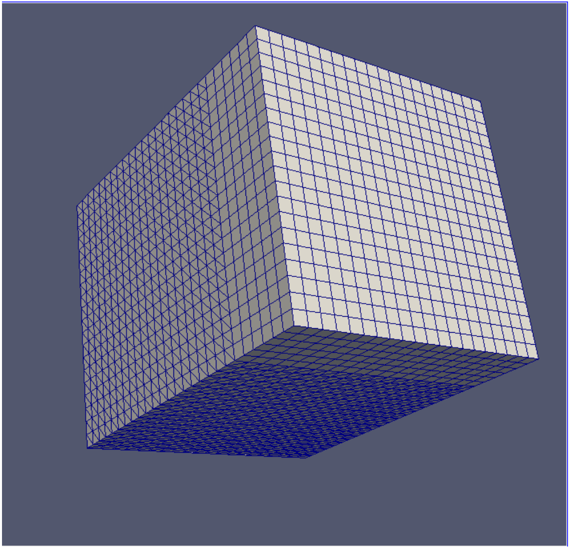
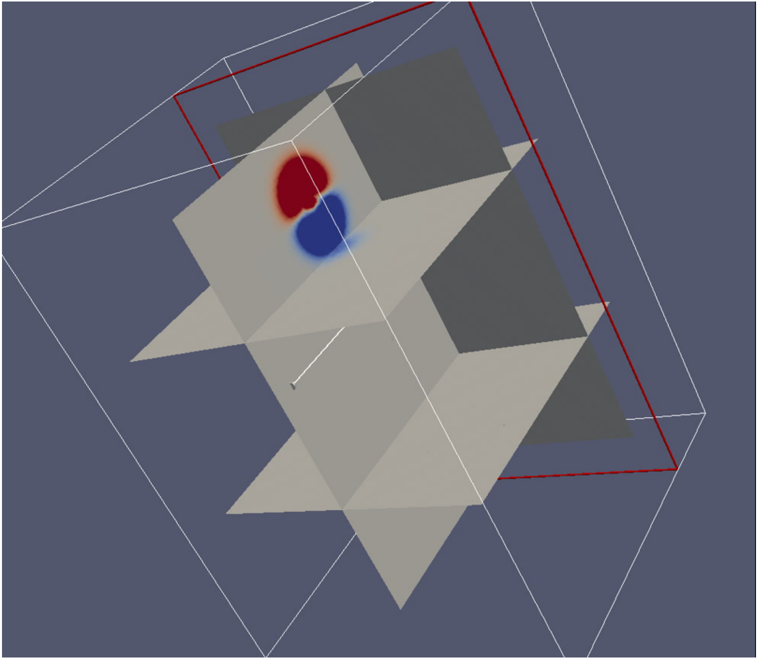
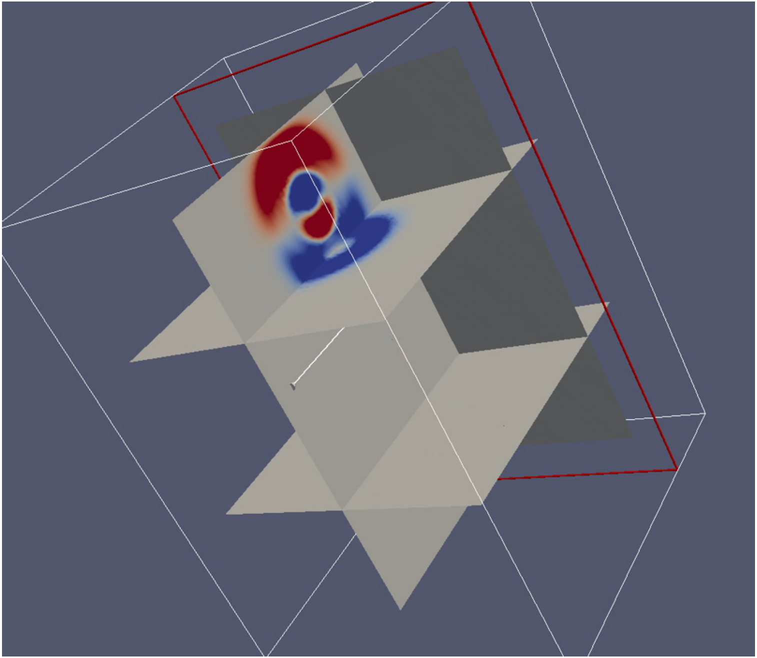
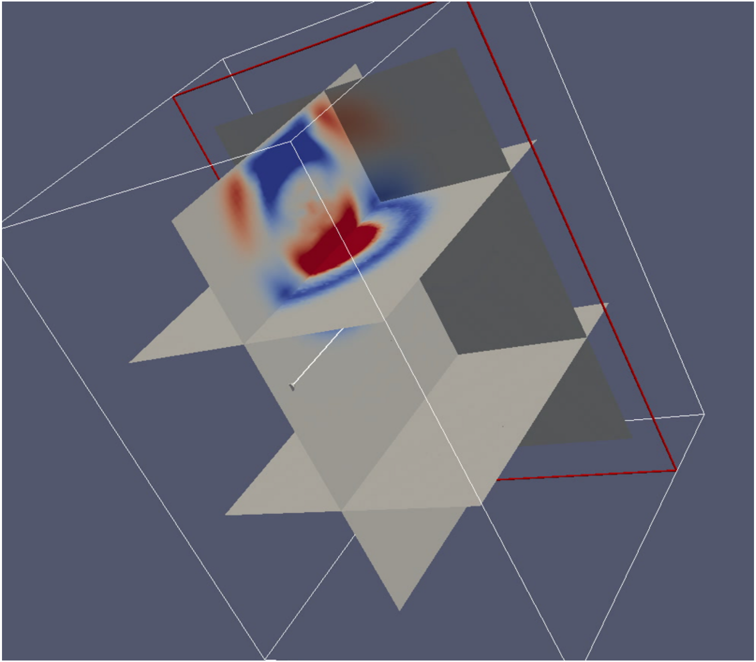
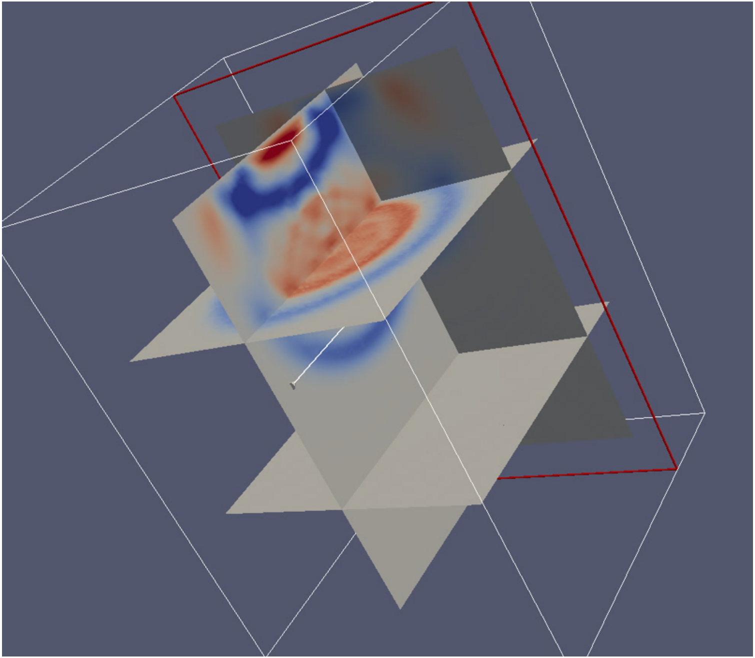
- Require introducing a new matrix which handles the rotation cases between two elements.

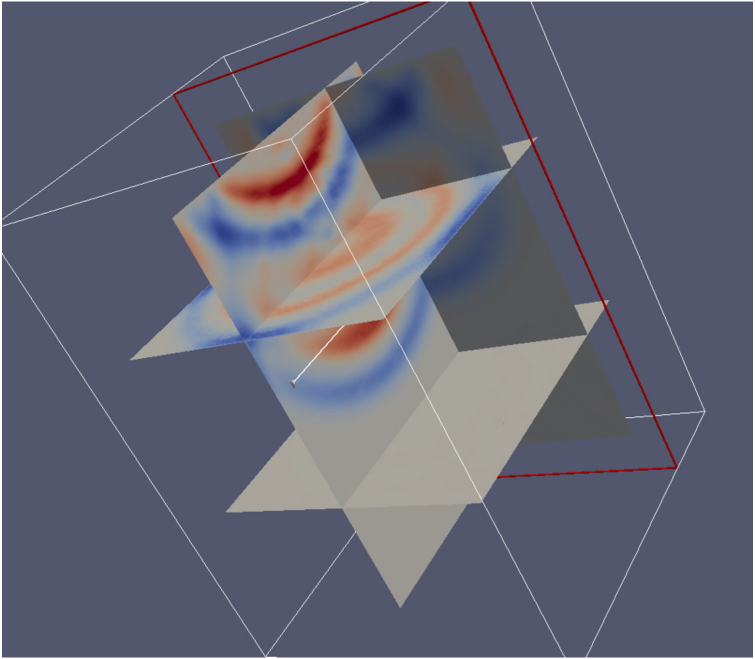


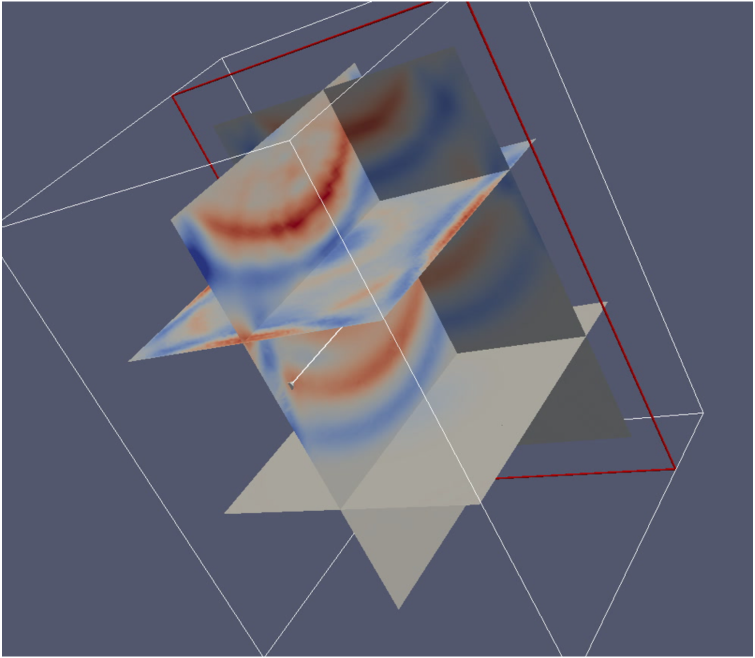


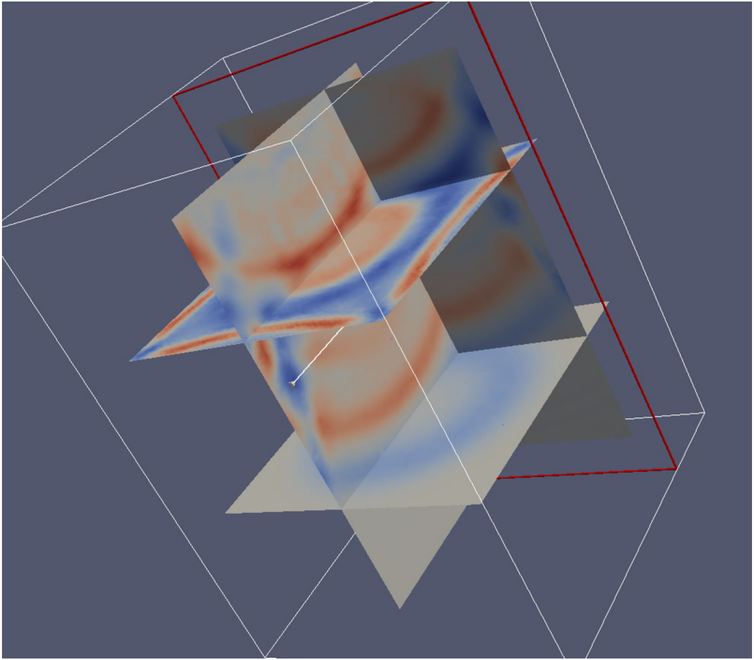












Conclusion and perspectives

Conclusion

1. SEM is more efficient on structured quadrangle mesh than DG
2. Build a variational formulation for DG/SEM coupling and find a CFL condition that ensures stability
3. Show the utility of using hybrid meshes and method coupling (reduce computational cost,...)

Perspectives

- Implement DG/SEM coupling on the code (2D) ✓
- Develop DG/SEM coupling in 3D ✓
- Develop PML in the hexahedral part
- Add a local time-stepping scheme

Thank you for your attention !

Questions?

Error-order graphic

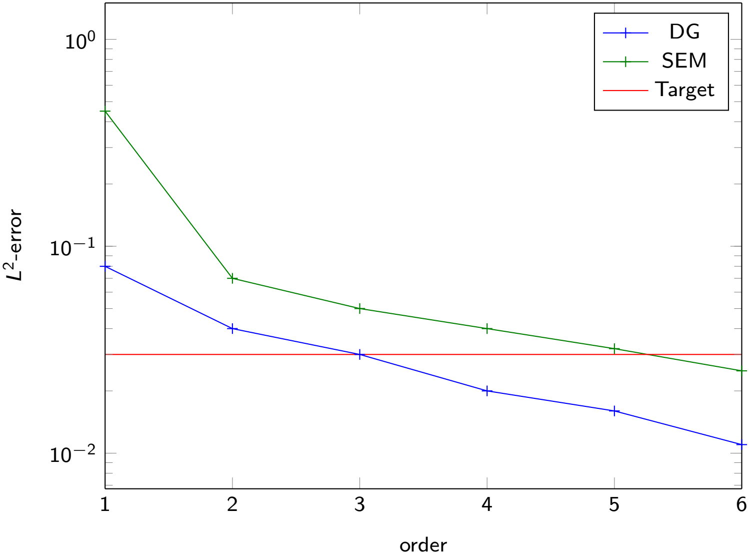


Figure: L^2-error comparison on a 10000 cells mesh

Error-order graphic

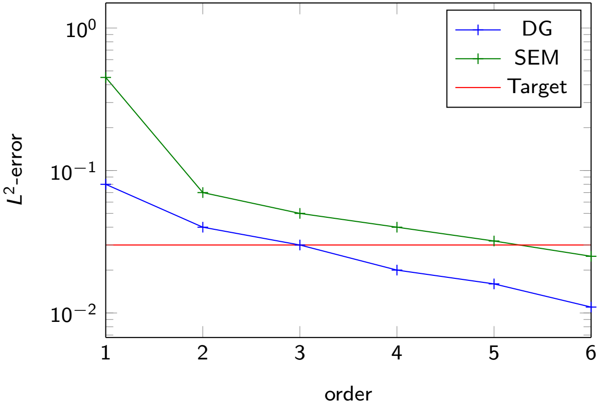


Figure: L^2-error comparison on a 10000 cells mesh

	CFL	L2-error	CPU-time	Nb of time steps
DG	2e-3	3e-2	7.93	1000
SEM(order five)	2.13e-3	3e-2	9.06	943

Figure: SEM and DG comparison with fixed error